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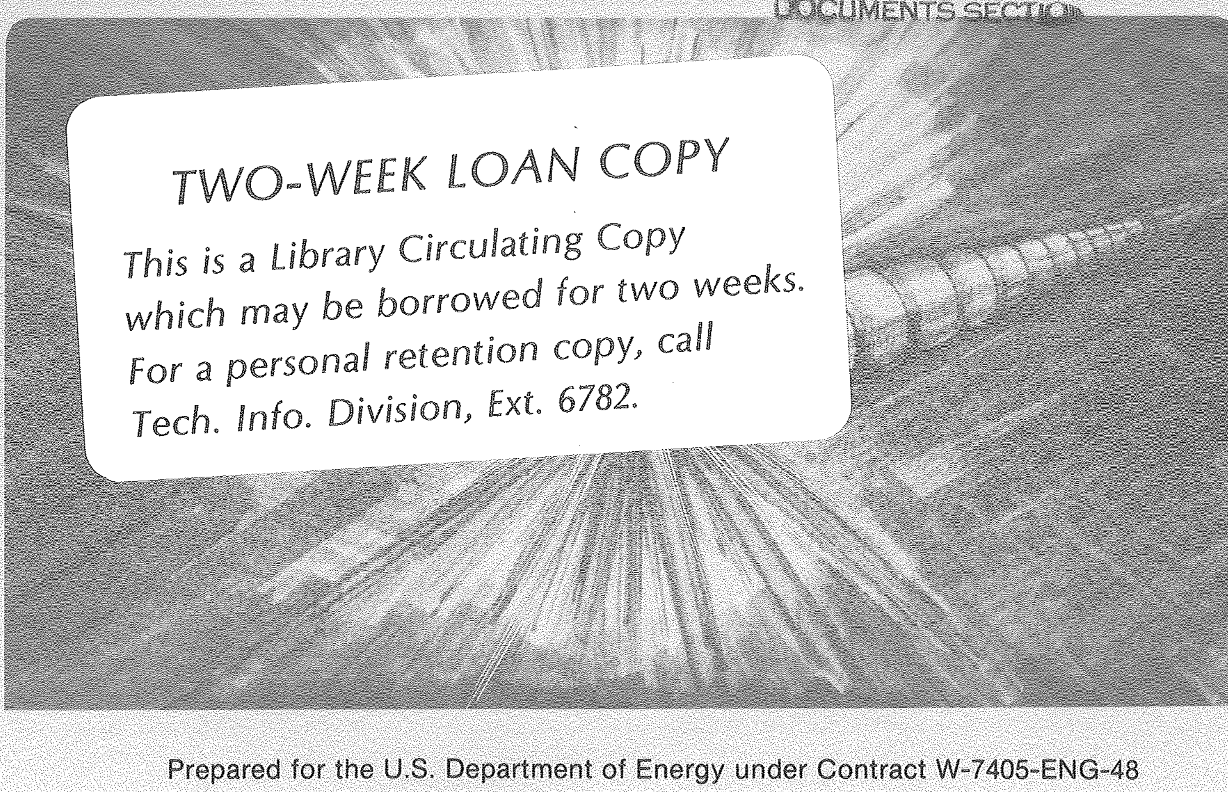

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Statistical Analysis of a Deterministic Stochastic Orbit*

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ABSTRACT

If the solution of a deterministic equation is stochastic (in the sense of orbital instability), it can be subjected to a statistical analysis. This is illustrated for a coded orbit of the Chirikov mapping. Statistical dependence and the Markov assumption are tested. The Kolmogorov-Sinai entropy is related to the probability distribution for the orbit.

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The idea that a deterministic equation can have a solution with random properties has encountered understandable resistance in the physics community. Our gradual acceptance of this idea is due in large part to the pioneering expositions of Chirikov[1] and of Ford,[2] who explained the concepts of the mathematicians, and presented numerical experiments and relatively simple analytic formulations. Most recently, a deeper understanding has come about from the studies of Shimada,[3] who performed a statistical analysis of a solution of the Lorenz equations.

In this paper, we devote ourselves to applying some standard methods of statistical analysis to a single numerical solution of a deterministic system that arises in several guises in plasma physics as well as in other applications. This is the Chirikov "standard mapping"[1]:

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$$I_{n+1} = I_n + \frac{K}{2\pi} \sin 2\pi\theta_n$$
$$\theta_{n+1} = \theta_n + I_{n+1} \qquad \text{(mod 1)}$$

which is area-preserving, and depends on the single parameter K. By a "solution," we mean the set $\{I_n, \theta_n\}$ $(n = 0, 1, 2, \ldots)$ for some chosen value of K and some initial condition (I_0, θ_0). It is known from numerical evidence[1] that for $K \gtrsim 5$ and for most initial conditions, the solution is "stochastic;" i.e., solutions with neighboring initial conditions diverge exponentially with n, at a rate given by the Lyapunov characteristic exponent[4] $\lambda \approx \ln \frac{K}{2}$. For stochastic solutions, the statistical properties we shall study are insensitive to numerical roundoff errors, since the latter cannot compete against the exponentiation.[5]

We begin by coarse-graining (or "partitioning") the phase-space $(0 \leq I < 1,\ 0 \leq \theta < 1)$. In practice, we simply take each I_n, calculated with reasonable precision, say $I_5 = .8279643\ldots$, and discard all but its first digit; i.e., we have $I_5 \rightarrow 8$ (ignoring the decimal point). Thus the numerical solution, $\{I_n, \theta_n\}$, is replaced by a sequence of digits, e.g., 3, 7, 4, 1, 5, 8, 6, 2, ...; the n^{th} digit labels which of ten strips in phase-space is occupied by the solution at the n^{th} iteration of the mapping. This sequence is called the "coded" solution or the "symbolic orbit."

We now look for aspects[5] of randomness among this sequence of digits. First we evaluate P(a), the relative frequency of the digit a, for each $a = 0,1,\ldots,9$. For an ergodic orbit, we expect $P(a) = 0.1$, for each a. This is indeed found to be true, for a stochastic solution.

Next, we examine $P_m(a,b)$, the relative frequency of the digit $\underline{a}$ followed $\underline{m}$ iterations later by the digit $\underline{b}$. If there were no short-range order in the set of digits, we would have $P_m(a, b) = P(a)P(b) = 0.01$. Statistical dependence is revealed by the ratio $P_m(a,b)/P(a)P(b)$ departing from one. In Figure 1, we consider this ratio for $b = 1$, and for $a = 0,1,\ldots,9$. For each $\underline{m}$, we plot the maximum and minimum with respect to $\underline{a}$ of this ratio. We see that the ratio approaches one, as $\underline{m}$ increases, indicating statistical independence after sufficiently many iterations. (The error-bar represents a naive estimate of the expected fluctuation in the measured ratio resulting from the finite number of iterations.) For higher K, the orbit is more stochastic (larger Lyapunov exponent), so statistical independence is achieved more rapidly.

A frequently-made assumption about a stochastic process is the Markov property, namely that the conditional probability $P_m(c;b,a)$ [probability of $\underline{c}$, given that $\underline{b}$ appeared $\underline{m}$ steps earlier, and that $\underline{a}$ appeared $\underline{m}$ further steps earlier] is independent of the earlier value $\underline{a}$. For Figure 2, we choose $c = 3$, $b = 5$, and plot the maximum and minimum with respect to $\underline{a}$ of the ratio $P_m(c;b,a)/P_m(c;b)$. We see non-Markovian behavior for $m < 5$, but for $m \gtrsim 5$, the Markov assumption appears valid.

Considering now only $m = 1$, we examine the sequence of joint probabilities $P(a) = 0.1$, $P(a,b)$, $P(a,b,c)$, $P(a,b,c,d),\ldots$, which represent the short-range order of the coded solution. For each P, we can evaluate the corresponding Gibbs-Shannon entropy $S_n \equiv -\frac{1}{n}\Sigma\, P\ell n\, P$, where n is here the number of successive digits in the argument of P.

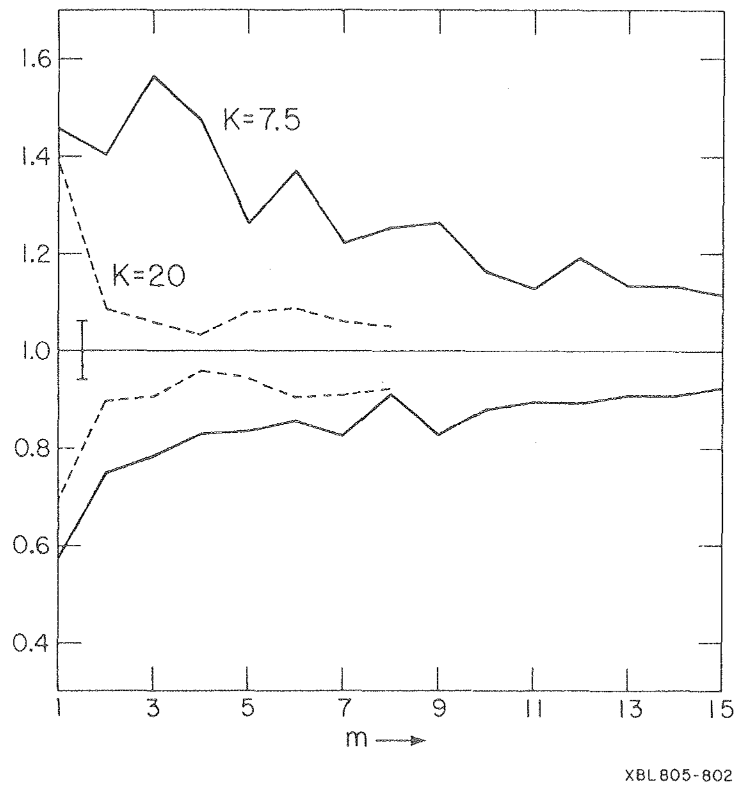

Fig. 1

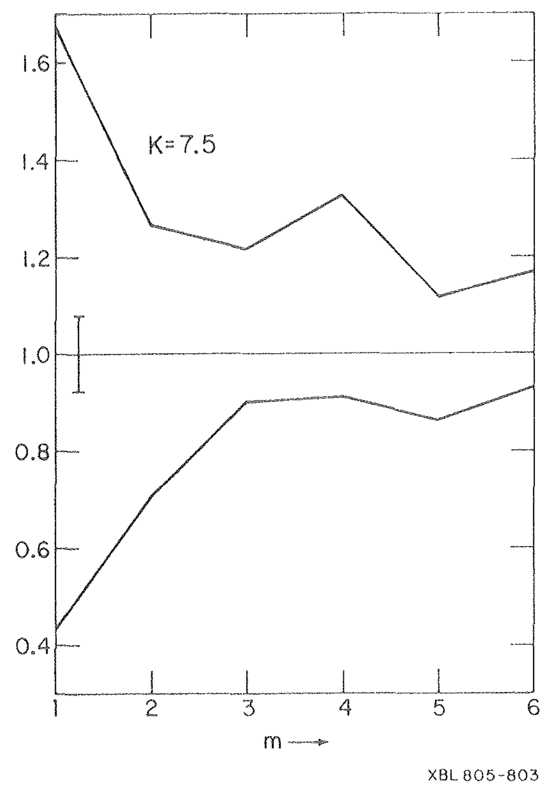

Fig. 2

(Thus S_n lies between zero, for perfect order, and $\ln 10$, for complete disorder.) As n increases, S_n rapidly approaches (from above) an asymptotic value S_∞. The latter still depends strongly on the coarse-graining scheme used, of which ours is an arbitrary one. In particular, it depends on the number of cells (in our case, 10) in the partition of phase space. As this number is increased, so does S_∞; remarkably, it has a finite limit, which is called the Kolmogorov-Sinai entropy,[6] and this limit is numerically equal to the Lyapunov exponent[4] averaged over initial conditions. This agreement has been verified by Shimada[3] for the Lorenz model; our attempts to verify similar agreement for the Chirikov mapping have so far been limited by computer time.

The results we have obtained appear to be consistent with the mathematical ideas expressed by Sinai[5]. However, no quantitative theory exists as yet, to our knowledge. We encourage our colleagues to develop such a theory.

We are indebted to Oscar Manley for providing us with a translation of Sinai's review, and to Oscar Lanford for providing us with mathematical advice.

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